Pergamon

PII: S0021-8928(98)00072-0

J. Appl. Maths Mechs, Vol. 62, No. 4, pp. 555–563, 1998

0021-8928/98/$—see front matter

GUARANTEED CONTROL IN DIFFERENTIAL GAMES WITH ELLIPSOIDAL PAYOFF†

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(*Received* 19 *February* 1997)

A class of antagonistic linear differential games (DGs) in a fixed time interval with ellipsoidal payoff functional is considered. This class of DGs includes problems which assume both rigid constraints on the players' controls and requirements to minimize control expenses. Other known classes of differential games, such as linear DGs with a quadratic performance index and linear DGs with ellipsoidal terminal sets and admissible sets of controls for the players, considered in Kurzhanskii's ellipsoidal technique, are limiting cases of DGs of this class. The concept of a u-strategic function, which expresses the property of u-stability for ellipsoidal functions, is introduced. An effective algorithm is presented for computing a u-strategic function, based on Kurzhanskii's ellipsoidal technique. The main result of this paper is that a guaranteed positional strategy for player $\mathbf{u}$ is defined by a certain explicit formula in terms of a u-strategic function. The proof of this result is based on a viability theorem for differential equations.

1. DIFFERENTIAL GAMES WITH ELLIPSOIDAL PAYOFF

Let PD^n denote the class of positive definite symmetric $n \times n$ matrices, and let Ω^n denote the following set of triples of a matrix and two numbers

$$\Omega^n = \left\{(K,\theta,\sigma) \middle| K \in \mathrm{PD}^n,\quad \theta > 0,\quad \sigma \in \mathbb{R}\right\}$$

For any triple $\omega = (K, \theta, \sigma) \in \Omega^n$, we define a function $\varphi(\cdot\,; \omega)$ of a vector variable $x \in \mathbb{R}^n$

$$\varphi(x;\omega) = \begin{cases} -\sigma - \theta\sqrt{1 - x^T K^{-1} x}, & x^T K^{-1} x \leqslant 1 \\ +\infty, & x^T K^{-1} x > 1 \end{cases}$$

is part of the surface of an ellipsoid in $\mathbb{R}^{n+1}$, and for that reason the functions $\varphi(\cdot\,; \omega)$ are called ellipsoidal functions.

We define the conjugate to a function f: $\mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ [1] as the function

$$f^*(\psi) = \sup_{x \in \mathbb{R}^n} \left(\psi^T x - f(x)\right)$$

Lemma 1.1. For any triple $\omega = (K, \theta, \sigma) \in \Omega^n$, the function $\varphi(\cdot\,; \omega)$ is convex and semicontinuous from below and its conjugate has the form

$$\varphi^*(\psi;\omega) = \sigma + \sqrt{\theta^2 + \psi^T K \psi}$$

We define the classes of ellipsoidal functions Φ^n and the conjugate ellipsoidal functions Φ^{n*} as follows:

$$\Phi^n = \left\{\varphi(\cdot;\omega) \middle| \omega \in \Omega^n\right\},\quad \Phi^{n*} = \left\{\varphi^*(\cdot;\omega) \middle| \omega \in \Omega^n\right\}$$

Consider the differential game (DG)

$$\dot{x}(t) = v(t) - u(t),\quad t \in [T_0, T] \tag{1.1}$$

with payoff functional

$$J = \alpha(x(T)) + \int_{T_0}^{T} \left(\beta(t,u(t)) - \gamma(t,v(t))\right) dt \tag{1.2}$$

†*Prikl. Mat. Mekh.* Vol. 62, No. 4, pp. 598–607, 1998.

where $x(t) \in \mathbb{R}^n$ is the phase vector of the system and $u(t), \upsilon(t) \in \mathbb{R}^n$ are the players' controls.

We will call $\alpha: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ the terminal function and $\beta, \gamma:[T_0, T] \times \mathbb{R} \cup \{+\infty\}$ the penalty functions. The aim of player **u** is to minimize the value of the functional J, while the aim of player **v** is to maximize it.

It is well known that a DG with linear dynamics

$$\dot{x}(t) = A(t)x(t) - B(t)u(t) + C(t)v(t)$$

may be reduced, by a suitable coordinate transformation, to the form of a game with dynamics (1.1).

We shall say that the payoff functional (1.2) is ellipsoidal if the terminal and penalty functions are ellipsoidal functions of the phase vector

$$\alpha, \beta(t,\cdot), \gamma(t,\cdot) \in \Phi^n \ \forall t \in [T_0, T] \tag{1.3}$$

This means that triples ω_α, $\omega_\beta(t)$, $\omega_\gamma(t) \in \Omega^n$ exist such that

$$\alpha(x) = \varphi(x;\omega_\alpha), \quad \beta(t,u) = \varphi(u;\omega_\beta(t))$$

$$\gamma(t,v) = \varphi(v;\omega_\gamma(t)), \quad \omega_\alpha = (K_\alpha, \theta_\alpha, \sigma_\alpha)$$

$$\omega_\beta(t) = (K_\beta(t), \theta_\beta(t), \sigma_\beta(t)), \quad \omega_\gamma(t) = (K_\gamma(t), \theta_\gamma(t), \sigma_\gamma(t))$$

Throughout this paper it will be assumed that the functions ω_β, ω_γ: $[T_0, T] \to \Omega^n$ are continuous.

Let $\delta(\cdot; M)$ denote the indicator function of a set $M \subset \mathbb{R}^n$

$$\delta(x;M) = \begin{cases} 0, & x \in M \\ +\infty, & x \notin M \end{cases}$$

In the limiting case, when $\theta_\alpha = \theta_\beta(t) = \theta_\gamma(t) = 0$, $\sigma_\alpha = \sigma_\beta(t) = \sigma_\gamma(t) = 0$, the terminal and penalty functions are the indicator functions of certain ellipsoids M, $P(t)$, $Q(t)$

$$\alpha(x) = \delta(x; M), \quad \beta(t,x) = \delta(x; P(t)), \quad \gamma(t,x) = \delta(x; Q(t))$$

In that case the DG becomes a game with terminal set M and constraints on the controls

$$u(t) \in P(t), \quad \upsilon(t) \in Q(t) \tag{1.4}$$

In many applications, it is not only important for player **u** to reach the terminal ellipsoid M but in addition it is desirable to minimize the distance to its centre. These requirements may be formalized by considering the DG (1.1), (1.2) with constraints (1.4) and terminal function $\alpha \in \Phi^n$: $\alpha(x) = \varphi(x; (K_\alpha, \theta_\alpha, \sigma_\alpha))$, where K_α is the matrix of the terminal ellipsoid $M = \{x \in \mathbb{R}^n | x^T K_\alpha^{-1} x \leq 1\}$ and θ_α is a scaling coefficient.

In practical problems, besides the given rigid constraints on the players' controls, it is often also necessary to minimize the costs of the controls. Let us assume that the set of admissible control vectors for each player at each instant of time is an ellipsoid. The costs for the player's control are minimal if the control vector coincides with the centre of the ellipsoid. These requirements may be formalized by introducing penalty functions $\beta(t, \cdot)$, $\gamma(t, \cdot) \in \Phi^n$.

The limit relations

$$\frac{1}{2}x^T Kx = \lim_{\theta \to +\infty} \varphi(x;(\theta K^{-1}, \theta, -\theta)) = \lim_{\theta \to +\infty} \varphi^*(x;(\theta K^{-1}, \theta, -\theta))$$

indicate that DGs with a quadratic performance index are a limiting case of differential games (1.1)–(1.3).

2. THE STRATEGIC FUNCTION

It is known from differential game theory [2] that a guaranteed strategy for player **u** can be constructed using a u-stable function. In this paper, a u-stable function will be sought in the class of ellipsoidal functions $\varphi(\cdot; \omega)$, $\omega \in \Omega^n$. In that connection, instead of the concept of u-stability for a function $\varphi(x; \omega(t))$, it will be more convenient to define a u-strategic function, as follows.

Definition 2.1. A continuously differentiable function ω: $[T_0, T] \to \Omega^n$ is said to be u-strategic in the

DG (1.1), (1.2) if $\omega(T) = \omega_\alpha$ and for all $t \in [T_0, T]$, $\psi \in \mathbb{R}^n$,

$$\partial\varphi^*(\psi;\omega(t))/\partial t \geq \gamma^*(t,\psi) - \beta^*(t,\psi) \tag{2.1}$$

where $\beta^*(t,\cdot)$, $\gamma(t,\cdot)$ are the conjugates of $\beta(t,\cdot)$, $\gamma(t,\cdot)$.

Using approximations of the ellipsoids as proposed in [3, 4], we obtain an effective method of computing u-strategic functions.

Lemma 2.1. Let $\lambda(t)$ be a real-valued function and let $S(t)$ be a matrix-valued function, both continuous in the interval $[T_0, T]$, $\lambda(t) > 0$, $S(t)$ being a non-singular $n \times n$ matrix. Let $\omega(t) = (K(t), \theta(t), \sigma(t)) \in \Omega^n$ be the continuously differentiable function defined by solving the following Cauchy problems

$$\dot\sigma(t) = \sigma_\gamma(t) - \sigma_\beta(t), \quad \sigma(T) = \sigma_\alpha$$

$$\dot\theta(t) = \frac{\theta_\gamma^2(t)}{2\lambda(t)\theta(t)} + \frac{\lambda(t)\theta(t)}{2} - \theta_\beta(t), \quad \theta(T) = \theta_\alpha$$

$$\dot K(t) = \frac{K_\gamma(t)}{\lambda(t)} + \lambda(t)K(t) - \left(S^{-1}\right)^T(t)\left(K_1(t) + K_1^T(t)\right)S^{-1}(t)$$

$$K_1(t) = \left(S^T(t)K(t)S(t)\right)^{1/2}\left(S^T(t)K_\beta(t)S(t)\right)^{1/2}, \quad K(T) = K_\alpha$$

Then $\omega(t)$ is a u-strategic function.

3. A GUARANTEED CONTROL THEOREM

Let dom f denote the effective set of the function f: $\mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$

$$\operatorname{dom} f = \left\{x \mid f(x) < +\infty\right\}$$

Definition 3.1. A positional strategy for player $\mathbf{u}$ is a function $u_{\text{pos}}(t,x)$ which is continuous in t, satisfies a Lipschitz condition in x and is such that $u_{\text{pos}}(t,x) \in \operatorname{dom}\beta(t,\cdot)$ for $t \in [T_0, T]$, $x \in \mathbb{R}^n$.

Definition 3.2. A function W: $[T_0, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called a guaranteed result for a positional strategy u_{pos} in the game (1.1), (1.2) if, for any $(t_0, x_0) \in \operatorname{dom} W$ and any absolutely integrable function υ: $[T_0, T] \to \mathbb{R}^n$, $\upsilon(t) \in \operatorname{dom}\gamma(t,\cdot)$

$$\alpha(x(T)) + \int_{t_0}^{T} \left(\beta\left(t, u_{\text{pos}}(t, x(t))\right) - \gamma(t, \upsilon(t))\right)dt \leq W(t_0, x_0) \tag{3.1}$$

where $x(t)$ is the solution of the equation $\dot x(t) = u_{\text{pos}}(t, x(t)) - \upsilon(t)$ with initial condition $x(t_0) = x_0$. The strategy u_{pos} thus defined is known as a guaranteed control for the result $W(t, x)$.

Let $x \in \mathbb{R}^n$, ω, $\omega_\beta \in \Omega^n$, $\omega = (K, \theta, \sigma)$, $\omega_\beta = (K_\beta, \theta_\beta, \sigma_\beta)$ be given. Define

$$u_0(x,\omega,\omega_\beta) = \begin{cases} \dfrac{\theta K_\beta K^{-1}x}{\sqrt{\theta_\beta^2(1 - x^T K^{-1}x) + \theta^2 x^T K^{-1}K_\beta K^{-1}x}}, & x^T K^{-1}x \leq 1 \\[2ex] \dfrac{K_\beta K^{-1}x}{\sqrt{x^T K^{-1}K_\beta K^{-1}x}}, & x^T K^{-1}x \geq 1 \end{cases} \tag{3.2}$$

Guaranteed control theorem. Let the function $\omega(t)$ be a u-strategy. Then the function

$$u_{\text{pos}}(t, x) = u_0(x, \omega(t), \omega_\beta(t)) \tag{3.3}$$

is a positional strategy for player $\mathbf{u}$, and the function

$$W(t, x) = \varphi(x; \omega(t)) \tag{3.4}$$

is a guaranteed result for that strategy.

Remark. An analogous construction yields a positional strategy and guaranteed result function for player v.

4. LOWER DERIVATIVE LEMMA

Let $\| K \|$ denote the maximum modulus of the eigenvalues of a matrix $K \in \mathbb{R}^{n \times n}$. Let $\omega = (K, \theta, \sigma) \in \mathbb{R}^{n \times n} \times \mathbb{R} \times \mathbb{R}$. Put

$$\|\omega\| = \|K\| + |\theta| + |\sigma| \tag{4.1}$$

We define the distance between the elements ω_1 and ω_2 of Ω^n as $\| \omega_1 - \omega_2 \|$.
The following lemma may be derived from formula (3.2).

Lemma 4.1. For any compact sets $\Omega, \Omega_\beta \subset \Omega^n$, the function $u_0(x, \omega, \omega_\beta)$ satisfies a Lipschitz condition on the set $\mathbb{R}^n \times \Omega \times \Omega_\beta$.
Let $\psi_0 \in \mathbb{R}$, $\psi \in \mathbb{R}^n$, $(\psi_0, \psi) \neq (0, 0)$, $\omega = (K, \theta, \sigma) \in \Omega^n$ be given. We define

$$q(\psi_0, \psi, \omega) = \frac{K\psi}{\sqrt{\theta^2 \psi_0^2 + \psi^T K \psi}} \tag{4.2}$$

Given $x \in \mathbb{R}^n$, $\omega = (K, \theta, \sigma) \in \Omega^n$ we define

$$\psi_0(x, \omega) = \begin{cases} \sqrt{1 - x^T K^{-1} x}, & x^T K^{-1} x \leq 1 \\ 0, & x^T K^{-1} x > 1 \end{cases}, \quad \psi(x, \omega) = \theta K^{-1} x$$

The following properties are immediate consequences of the definitions.

Property 4.1. For any $\omega \in \Omega^n, x \in \operatorname{dom} \varphi(\cdot; \omega)$, the vector $(\psi(x, \omega), \psi_0(x, \omega)) \in \mathbb{R}^{n+1}$ is an outward normal of the convex set $\operatorname{epi} \varphi(\cdot; \omega) = \{(\xi, \lambda) \in \mathbb{R}^{n+1} | \varphi(x; \omega) \geq \lambda\}$ at the point $(x, \varphi(x; \omega)) \in \operatorname{epi} \varphi(\cdot; \omega)$.

Property 4.2. For any $\omega \in \Omega^n$, $\psi_0 \in \mathbb{R}$, $\psi \in \mathbb{R}^n$, where $(\psi_0, \psi) \neq (0, 0)$, the vector $q(\psi_0, \psi, \omega)$ is the unique solution of the minimization problem

$$\min_{x \in \operatorname{dom} \varphi(\cdot; \omega)} \left(\psi_0 \varphi(x; \omega) - \psi^T x \right)$$

Property 4.3. For any $x \in \mathbb{R}^n$, $\omega\ \omega_\beta \in \Omega^n$, we have the equality $u_0(x, \omega, \omega_\beta) = q(\psi_0(x, \omega), \psi(x, \omega), \omega_\beta)$.
It follows from Lemma 4.1, and also from Properties 4.2 and 4.3, that the function $u_{\mathrm{pos}}(t, x)$ of (3.3) is a positional strategy in the sense of Definition 3.1.
Using Properties 4.1–4.3 we can prove the following.

Lemma 4.2. For any compact sets $\Omega, \Omega_\beta \subset \Omega^n$, numbers $\delta_0 > 0$, $C_u \in \mathbb{R}$ exist such that, for any $\delta \in (0, \delta_0)$, $z \in \operatorname{dom} \varphi(\cdot; \omega) + \delta \operatorname{dom} \varphi(\cdot; \omega_\beta)$

$$\min_{u \in \mathbb{R}^n} \left(\varphi(z - \delta u; \omega) + \delta \varphi(u; \omega_\beta) \right)$$

is attained on a unique vector u_δ, where

$$\left| u_\delta - u_0\left(z, \omega, \omega_\beta\right) \right| \leq C_u \delta$$

Lemma 1.1 and formula (4.1) imply the following.

Lemma 4.3. For any compact set $\Omega \subset \Omega^n$, a number C_Ω exists such that, for any $\omega_1, \omega_2 \in \Omega$, $\psi \in \mathbb{R}^n$

$$\left| \varphi^*(\psi; \omega_1) - \varphi^*(\psi; \omega_2) \right| \leq C_\Omega \|\omega_1 - \omega_2\| (1 + |\psi|)$$

The following lemma is an immediate corollary of the definition of a conjugate function.

Lemma 4.4. 1. Suppose we are given the functions $f_1, f_2: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and

$$f(x) = \inf_{z \in \mathbb{R}^n} \left(f_1(x-z) + f_2(z) \right)$$

Then $f^*(\psi) = f_1^*(\psi) + f_2^*(\psi)$.

2. Suppose we are given the functions $f, g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and let $f^*(\psi) < +\infty \forall \psi \in \mathbb{R}^n$. Then

$$\left(g - f^*\right)^*(x) = \sup_{y \in \operatorname{dom} f} \left(g^*(x+y) - f(y) \right)$$

3. For any given function $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and number $\delta > 0$, we have $(\delta g)^*(x) = \delta g^*(x/\delta)$.

The following lemma is derived from Lemma 4.4.

Lemma 4.5. Suppose we are given the convex semicontinuous from below functions g_0, g_1, g_2, g_3: $\mathbb{R}^n \to \mathbb{R}$ and a number $\delta > 0$. Then

$$\left(g_0 + \delta(g_1 + g_2 - g_3)\right)^*(x) = \sup_{v \in \operatorname{dom} g_3^*} \inf_{y \in \mathbb{R}^n} \inf_{u \in \mathbb{R}^n} \Big(g_0^*(x + \delta(v - y - u)) +$$

$$+\delta(g_1^*(u) + g_2^*(y) - g_3^*(v)) \Big)$$

For any vector $l \in \mathbb{R}^n$, we define the lower derivative [5] of a function $W: [T_0, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ in the direction $(1, l)$ at a point $(t, x) \in \operatorname{dom} W$

$$\partial^-_{(1,l)} W(t,x) = \liminf_{\substack{s \to l \\ \delta \to +0}} \frac{W(t+\delta, x+\delta s) - W(t,x)}{\delta}$$

Lemma 4.6. Let $\omega(t)$ be a u-strategic function, and let the functions $u_{\text{pos}}(t,x)$, $W(t,x)$ be defined by formulae (3.3) and (3.4). Let $(t,x) \in \operatorname{dom} W$, $\upsilon \in \operatorname{dom} \gamma(t, \cdot)$. Then

$$\partial^-_{(1,t)} W(t,x) + \beta(t, u_{\text{pos}}(t,x)) - \gamma(t, \upsilon) \leq 0, \text{ where } 1 = \upsilon - u_{\text{pos}}(t,x)$$

Proof. Let $W^*(t, \cdot)$. By Definition 2.1 and formula (3.4), for sufficiently small $\delta > 0$

$$W^*(t,\psi) - W^*(t+\delta,\psi) \leq \int_t^{t+\delta} \left(\beta^*(\tau,\psi) - \gamma^*(\tau,\psi) \right) d\tau$$

It follows from the continuity of $\omega_\beta(t)$ and $\omega_\psi(t)$ and from Lemma 4.3 that a function $\varepsilon_1(\delta) \to 0$ $(\delta \to +0)$ exists such that

$$W^*(t,\psi) - W^*(t+\delta,\psi) \leq \delta\left(\beta^*(t,\psi) - \gamma^*(t,\psi)\right) + \delta\varepsilon_1(\delta)\left(1 + |\psi|\right)$$

Apply Lemma 4.5 for the functions

$$g_0(\psi) = W^*(t+\delta,\psi), \quad g_1(\psi) = \beta^*(t,\psi)$$
$$g_2(\psi) = \varepsilon_1(\delta)|\psi|, \quad g_3(\psi) = \gamma^*(t,\psi)$$

Then $g_2^*(y) = 0$ for $|y| \leq \varepsilon_1(\delta)$, $g_2^*(y) = +\infty$ for $|y| > \varepsilon_1(\delta)$ and

$$W(t,x) = W^{**}(t,x) \geq \left(g_0 + \delta(g_1 + g_2 - g_3)\right)^*(x) - \delta\varepsilon_1(\delta) =$$
$$= \sup_{v \in \operatorname{dom}\gamma(t,\cdot)} \inf_{|y| \leq \varepsilon_1(\delta)} \inf_{u \in \mathbb{R}^n} \left(W(t+\delta, x+\delta(v-y-u)) + \delta\beta(t,u) - \delta\gamma(t,v) \right) - \delta\varepsilon_1(\delta)$$

We fix an arbitrary $\upsilon \in \operatorname{dom} \gamma(t, \cdot)$.

Since $W(t,x) < +\infty$, it follows that

$$\inf_{|y| \leq \varepsilon_1(\delta)} \inf_{u \in \mathbb{R}^n} \left(W(t+\delta, x+\delta(v-y-u)) + \delta\beta(t,u) \right) < +\infty$$

and, since the functions $W(t+\delta, \cdot)$, $\beta(t, \cdot)$ are convex and semicontinuous from below, the infima are attained on certain vectors y_δ and u_δ. Apply Lemma 4.2 for $\omega = \omega(t+\delta)$, $\omega_\beta = \omega_\beta(t)$, $z = x + \delta(\upsilon - y_\delta)$. Then

$$\left|u_\delta - u_0\left(x+\delta\left(v-y_\delta\right), \omega(t+\delta), \omega_\beta(t)\right)\right| \le C_u\delta$$

Put $u_0 = u_0(x, \omega(t), \omega_\beta(t)) = u_{pos}(t,x)$. By Lemma 4.1 and the continuity of $\omega(t)$, a function $\varepsilon_2(\delta) \to 0$ ($\delta \to +0$), independent of υ, exists such that

$$\left|u_\delta - u_0\right| \le \varepsilon_2(\delta)$$

Since $u_\delta, u_0 \in \operatorname{dom} \beta(t, \cdot)$, it follows from the continuity of the function $\beta(t, \cdot)$ in its effective set that a function $\varepsilon_3(\delta) \to 0$ ($\delta \to +0$) exists for which

$$\left|\beta(t,u_\delta) - \beta(t,u_0)\right| \le \varepsilon_3(\delta)$$

Therefore

$$W(t,x) \ge W(t+\delta, x+\delta(v-y_\delta-u_\delta)) + \delta\beta(t,u_0) - \delta\gamma(t,v) - \delta(\varepsilon_1(\delta)+\varepsilon_3(\delta))$$

Choose some sequence $\delta_k \to +0$ and define $l_k = \upsilon - u_{\delta_k} - y_{\delta_k}$. Then

$$\left|l_k - l\right| = \left|u_0 - u_{\delta_k} - y_{\delta_k}\right| \le \varepsilon_1(\delta_k) + \varepsilon_2(\delta_k) \to 0$$

$$\left(W\left(t+\delta_k, x+\delta_k l_k\right) - W(t,x)\right)/\delta_k + \beta\left(t,u_0\right) - \gamma(t,v) \le \varepsilon_1\left(\delta_k\right) + \varepsilon_3\left(\delta_k\right) \to 0$$

and this completes the proof of the lemma.

5. PROOF OF THE GUARANTEED CONTROL THEOREM

The tangent cone to the set $M \subset \mathbb{R}^n$ at a point $z \in M$ is defined as

$$T_M(z) = \left\{ f \in \mathbb{R}^n \,\middle|\, \liminf_{\delta \to +0} \frac{\operatorname{dist}(z+\delta f; M)}{\delta} = 0 \right\} \tag{5.1}$$

where $\operatorname{dist}(x; M) = \inf_{y \in M} |x - y|$ is the distance from the point x to the set M.

We shall need the following theorem, known as the viability theorem [6, 7].

Viability theorem. Let $M \subset \mathbb{R}^n$ be a closed set and let f: $M \to \mathbb{R}^m$ be a continuous function. Assume that

$$f(z) \in T_M(z) \; \forall z \in M \tag{5.2}$$

Then, for any $z_0 \in M$, a number $\tau_0 > 0$ exists such that a solution in the interval $[0, \tau_0]$ of the equation $\dot{z}(\tau) = f(z(\tau))$ satisfying the initial condition $z(0) = z_0$, and moreover $z(\tau) \in M$ for any $\tau \in [0, \tau_0]$.

Lemma 5.1. Let the functions $u_{pos}(t,x)$ and $W(t,x)$ be defined by formulae (3.3) and (3.4). Suppose we are given a number $t_0 \in [T_0, T)$ and a vector $x_0 \in \operatorname{dom} W(t_0, \cdot)$. Then, for any continuous function υ: $[t_0, T] \to \mathbb{R}^n$ such that $(t, \upsilon(t)) < +\infty \; \forall t \in [t_0, T]$, and for any number $t_1 \in [t_0, T)$

$$W\left(t_1, x\left(t_1\right)\right) + \int_{t_0}^{t_1} \left(\beta\left(t, u_{pos}(t, x(t))\right) - \gamma(t, v(t))\right) dt \le W\left(t_0, x_0\right) \tag{5.3}$$

where $x(t)$ is a solution of the differential equation $\dot{x}(t) = \upsilon(t) - u_{pos}(t, x(t))$ satisfying the initial condition $x(t_0) = x_0$.

Proof. Fix arbitrary functions $\upsilon(t)$ and $x(t)$, satisfying the assumptions of the lemma. Let t_1^{max} denote the maximum number $t_1^* \in [t_0, T]$ such that inequality (5.3) holds for any $t_1 \in [t_0, t_1^*]$. Since $W(t,x)$ is semicontinuous from below, the maximum exists. Suppose that $t_1^{max} < T$. Define the set

$$M = \left\{ z = (t,x,y) \middle| t \in [t_0, T], x \in \mathbf{R}^n, y \in \mathbf{R}, W(t,x) + y \le W(t_0, x_0) \right\}$$

and a function f: $[t_0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$

$$f(t,x,y) = \left(T-t, (T-t)(v(t) - u_{pos}(t,x)), \quad (T-t)\left(\beta\left(t, u_{pos}(t,x)\right) - \gamma(t, v(t))\right)\right)$$

We will show that condition (5.2) holds.

Let $z = (t, x, y) \in M$. If $t = T$, then $f(z) = 0 \in T_M(z)$. Consider the case $t < T$.

Put $g = \beta(t, u_{pos}(t, x)) - \gamma(t, \upsilon(t))$. By Lemma 4.6, sequences of numbers $\delta_k \to +0$ and vectors $l_k \to l = \upsilon(t) - u_{pos}(t, x)$ exist such that

$$\liminf_{k\to\infty} \frac{W(t+\delta_k, x+\delta_k l_k) - W(t,x)}{\delta_k} + g \leqslant 0$$

Define

$$\Delta_k = \delta_k / (T-t), \quad t_k = t+\delta_k, \quad x_k = x+\delta_k l_k$$

$$y_k = y + \min\{W(t,x) - W(t_k, x_k), \delta_k g\}, \quad z_k = (t_k, x_k, y_k)$$

Then $W(t_k, x_k) + y_k \leqslant W(t, x) + y \leqslant W(t_0, x_0)$. Consequently, for sufficiently large k such that $t + \delta_k < T$, we have $z_k \in M$. By definition (5.1), the truth of condition (5.2) now follows from the relationships

$$\liminf_{k\to\infty} \left|z + \Delta_k f(z) - z_k\right| / \Delta_k \leqslant \liminf_{k\to\infty} \left((T-t)|l - l_k| + \left|\delta_k g - \min\{W(t,x) - W(t_k, x_k), \delta_k g\}\right| / \Delta_k\right) =$$

$$= (T-t) \liminf_{k\to\infty} \left(\max\left\{\left(W(t_k, x_k) - W(t,x)\right)/\delta_k + g, 0\right\}\right) = 0$$

Define the functions

$$y(t) = \int_{t_0}^{t} \left(\beta\left(\xi, u_{pos}(\xi, x(\xi))\right) - \gamma(\xi, \upsilon(\xi))\right) d\xi, \quad t(\tau) = T - \left(T - t_1^{\max}\right)e^{-\tau}$$

Note that $\dot{t}(\tau) = T - t(\tau)$ and the function $f(z)$ satisfies a Lipschitz condition. Hence it follows that $z(\tau) = (t(\tau), x(t(\tau)), y(t(\tau)))$ is the unique solution of the equation $\dot{z}(\tau) = f(z(\tau))$, satisfying the initial condition $z(0) = (t_1^{\max}, x(t_1^{\max}), y(t_1^{\max}))$. By the viability theorem, $\tau_0 > 0$: $z(\tau) \in M$ exists for $t \in [0, \tau_0]$, that is, inequality (5.3) is true for any $t_1 \in [t_1^{\max}, t(\tau_0)]$, contrary to the definition of $t_1^{\max}$. This contradiction shows that the assumption $t_1^{\max} < T$ cannot hold; consequently, $t_1^{\max} = T$, which completes the proof of the lemma.

To complete the proof of the guaranteed control theorem, it will suffice to show that Lemma 5.1 remains valid if the continuity condition for $\upsilon(t)$ is replaced by the requirement that the same function be absolutely integrable.

Given points $\tau_0, \tau_1, T_0 \leqslant \tau_0 < \tau_1 \leqslant T$ and a vector $\bar{\upsilon} \in \mathbb{R}^n$, we define

$$\bar{\gamma}(\bar{\upsilon}, \tau_0, \tau_1) = \inf_{\upsilon(\cdot)} \int_{\tau_0}^{\tau_1} \gamma(t, \upsilon(t)) dt \tag{5.4}$$

where the infimum is evaluated over all absolutely integrable functions υ: $[\tau_0, \tau_1] \to \mathbb{R}^n$ such that

$$\gamma(t, \upsilon(t)) < +\infty \quad \forall t \in [\tau_0, \tau_1], \quad \int_{\tau_0}^{\tau_1} \upsilon(t) dt = \bar{\upsilon}$$

It is easy to see that the function $\bar{\gamma}(\cdot, \tau_0, \tau_1)$ is convex. Applying the separation theorem to its epigraph, we obtain the following.

Lemma 5.2. Suppose we are given the points $\tau_0, \tau_1, T_0 \leqslant \tau_0 < \tau_1 \leqslant T$ and a vector $\bar{\upsilon}_0 \in \mathbb{R}^n$ such that $\bar{\gamma}(\bar{\upsilon}_0, \tau_0, \tau_1) < +\infty$. Then a number $\psi_0 \geqslant 0$ and a vector $\psi \in \mathbb{R}^n$, $(\psi_0, \psi) \neq (0, 0)$ exists, such that, for any $\bar{\upsilon} \in \bar{\gamma}(\cdot, \tau_0, \tau_1)$,

$$\psi_0 \bar{\gamma}(\bar{\upsilon}_0, \tau_0, \tau_1) - \psi^T \bar{\upsilon}_0 \leqslant \psi_0 \bar{\gamma}(\bar{\upsilon}_0, \tau_0, \tau_1) - \psi^T \bar{\upsilon}$$

Lemma 5.3. Let $T_0 \leqslant \tau_0 < \tau_1 \leqslant T$ and let $\upsilon_1(t)$ be a given function, absolutely integrable over the interval $[\tau_0, \tau_1]$, such that $\gamma(t, \upsilon_1(t)) < +\infty$ for $t \in [\tau_0, \tau_1]$. Then a continuous function υ_2: $[\tau_0, \tau_1] \to \mathbb{R}^n$ exists such that

$$\int_{\tau_0}^{\tau_1} \upsilon_2(t)dt = \int_{\tau_0}^{\tau_1} \upsilon_1(t)dt, \quad \int_{\tau_0}^{\tau_1} \gamma(t,\upsilon_2(t))dt \leq \int_{\tau_0}^{\tau_1} \gamma(t,\upsilon_1(t))dt \tag{5.5}$$

Proof. Define

$$\bar{\upsilon}_0 = \int_{\tau_0}^{\tau_1} \upsilon_1(t)dt \tag{5.6}$$

Since $\gamma(t, \upsilon(t)) \leq C < +\infty$, it follows that $\bar{\gamma}(\bar{\upsilon}_0, \tau_0, \tau_1) < +\infty$ and, by Lemma 5.2, a vector $(\psi_0, \psi) \in \mathbb{R} \times \mathbb{R}^n \backslash (0, 0)$ exists such that

$$\psi_0\bar{\gamma}(\bar{\upsilon}_0,\tau_0,\tau_1) - \psi^T\bar{\upsilon}_0 = \inf_{\bar{\upsilon} \in \operatorname{dom}\bar{\gamma}(\cdot,\tau_0,\tau_1)} (\psi_0\bar{\gamma}(\bar{\upsilon},\tau_0,\tau_1) - \psi^T\bar{\upsilon}) = \tag{5.7}$$

$$= \int_{\tau_0}^{\tau_1} \left(\inf_{\upsilon \in \operatorname{dom}\gamma(t,\cdot)} (\psi_0\gamma(t,\upsilon) - \psi^T\upsilon) \right) dt$$

It has been shown (see, e.g. [8]) that the infimum in (5.4) is attained. Let $\upsilon_0(t)$ denote the absolutely integrable function for which the minimum is attained in (5.4), and such that

$$\int_{\tau_0}^{\tau_1} \upsilon(t)dt = \bar{\upsilon}_0, \quad \gamma(t, \upsilon(t)) < +\infty$$

We put $\upsilon_2(t) = q(\psi_0, \psi, \omega_\gamma(t))$, where the function $q(\psi_0, \psi_0, \omega)$ is defined by (4.2). If $\upsilon_0(t) \neq \upsilon_2(t)$ on a set of non-zero measure, the, by Property 4.2

$$\psi_0\gamma(t, \upsilon_2(t)) - \psi^T\upsilon_2(t) < \psi_0\gamma(t, \upsilon_0(t)) - \psi^T\upsilon_0(t)$$

on that set. Then

$$\int_{\tau_0}^{\tau_1} \left(\psi_0\gamma(t, \upsilon_2(t)) - \psi^T\upsilon_2(t)\right)dt < \int_{\tau_0}^{\tau_1} \left(\psi_0\gamma(t, \upsilon_0(t)) - \psi^T\upsilon_0(t)\right)dt = \psi_0\bar{\gamma}(\bar{\upsilon}_0,\tau_0,\tau_1) - \psi^T\bar{\upsilon}_0$$

which contradicts (5.7). Consequently, $\upsilon_0(t) = \upsilon_2(t)$ almost everywhere. Taking (5.6) into account, as well as the definition of $\upsilon_0(t)$, we obtain (5.5).

Lemma 5.4. Let $t_0 \in [T_0, T]$ and let υ: $[t_0, T] \to \mathbb{R}^n$ be a given absolutely integrable function such that $\gamma(t, \upsilon(t)) < +\infty$. Then a sequence of piecewise-continuous functions υ_k: $[t_0, T] \to \mathbb{R}^n$ exists such that

$$\int_{t_0}^{T} \gamma(t, \upsilon_k(t))dt \leq \int_{t_0}^{T} \gamma(t, \upsilon(t))dt \tag{5.8}$$

and the solutions $x(t)$ and $x_k(t)$ of the equations $x(t) = \upsilon(t) - u_{\text{pos}}(t, x(t))$, $\dot{x}_k(t) = \upsilon(t) - u_{\text{pos}}(t, x_k(t))$, with initial data $x(t_0) = x_k(t_0) = x_0$, are such that

$$\max_{t\in[t_0,T]} |x_k(t) - x(t)| \to 0 \quad \text{as} \quad k \to \infty \tag{5.9}$$

Proof. Partition the interval $[t_0, T]$ into k equal subintervals by the points $t_i = t_0 + i(T - t_0)/k$.
By Lemma 5.3 ,applied to the interval $[\tau_0, \tau_1] = [t_i, t_{i+1}]$, a continuous function υ_k: $[t_i, t_{i+1}] \to \mathbb{R}^n$ exists such that

$$\int_{t_i}^{t_{i+1}} \upsilon_k(t)dt = \int_{t_i}^{t_{i+1}} \upsilon(t)dt, \quad \int_{t_i}^{t_{i+1}} \gamma(t,\upsilon_k(t))dt \leq \int_{t_i}^{t_{i+1}} \gamma(t,\upsilon(t))dt$$

Applying Lemma 5.3 to all the intervals $[t_i, t_{i+1}]$ $(i = 0, \ldots, k-1)$, we obtain a function $\upsilon_k(t)$, defined in $[t_0, T]$, which satisfies inequality (5.8).
Let $x(t)$ and $x_k(t)$ be as defined in the assumptions of the lemma. Using the fact that $u_{\text{pos}}(t, x)$ satisfies a Lipschitz condition with respect to x, we can readily prove (5.9).

Completion of the proof of the guaranteed control theorem.

We fix a point $t_0 \in [T_0, T]$, a vector $x_0 \in \operatorname{dom} W(t_0, \cdot)$ and an absolutely integrable function $\upsilon: [t_0, T] \to \mathbb{R}^n$ such that $\gamma(t, \upsilon(t)) < +\infty$ for all $t \in [t_0, T]$.

Applying Lemma 5.4, we obtain a sequence of piecewise-continuous functions $\upsilon_k: [t_0, T] \to \mathbb{R}^n$ for which relationships (5.8) and (5.9) hold.

For each $k = 1, 2, \dots$, applying Lemma 5.1 in each subinterval $[t_i, t_{i+1}]$ in which $\upsilon_k(t)$ is continuous, we obtain the inequalities

$$W(t_{i+1}, x_k(t_{i+1})) + \int_{t_i}^{t_{i+1}} \Big(\beta\big(t, u_{\text{pos}}(t, x_k(t))\big) - \gamma(t, v_k(t))\Big)dt \leq W(t_i, x_k(t_i))$$

which, taking into account the condition $W(T, x) = \alpha(x)$ (see (3.4)), gives

$$\alpha(x_k(T)) + \int_{t_0}^{T} \Big(\beta\big(t, u_{\text{pos}}(t, x_k(t))\big) - \gamma(t, v_k(t))\Big)dt \leq W(t_0, x_0)$$

Hence, by (5.8) and (5.9), the semicontinuity of α from below and the uniform continuity of the function $\beta(t, u_{\text{pos}}(t, \cdot))$, we obtain inequality (3.1).

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Translated by D.L.